PROBLEMS IN MEASURE THEORY

MANJUNATH KRISHNAPUR

Note: These are largely problems I put out when teaching the course on measure and integration at IISc during Jan-Apr 2018 and updated in 2021. While a few are made up, most are taken from various well-known books on the subject, for example, Bogachev, Cohn, Dudley, Rudin, Wheeden and Zygmund, Stein and Shakarchi etc.. Also I borrowed freely from Kavi Ramamurthy's problem sets (`http://www.isibang.ac.in/~adean/infsys/database/index.html`)

Do write to me if you spot any mistakes or if you have any suggestions. A few questions appear in more than one problem which happened because problems are collected from homeworks and exams, but I would like to eliminate exact duplication, if any.

Problem 1. In each of the following cases, find $\sigma(S)$ and $\mathcal{A}(S)$ (the sigma-algebra and algebra generated by S).

(1) X is a set and S is the collection of all singleton subsets of X.

(2) X is a set and S is the collection of all two-element subsets of X.

(3) $A_1, A_2, \ldots$ are pairwise disjoint sets of X such that $\bigcup\limits_n A_n = X$.

(4) Do the previous exercise if A_i are pairwise disjoint but their union may not equal X.

Problem 2. Let S be a collection of subsets of X that contains the empty set and whole set. Throw in the complements of all sets in S to get a larger collection S'. Take all finite intersections of elements of S' to get S''. Take all finite unions of elements of S'' to get $\mathcal{A}$. Show that $\mathcal{A}$ is the algebra generated by S.

Problem 3. Let $\mathcal{F}$ be a sigma algebra on X. Let $A_1, A_2, \ldots$ be elements of $\mathcal{F}$. Show that the following sets are also in $\mathcal{F}$ (first express the set in proper notation using unions, intersections, complements).

(1) The set of $x \in X$ that belong to exactly five of the A_ns.

(2) The set of $x \in X$ that belong to all except five of the A_ns.

(3) The set of $x \in X$ that belong to infinitely many of the A_ns.

(4) The set of $x \in X$ that belong to all but finitely many of the A_ns.

Problem 4. Decide whether the following statements are true or false and justify your answer.

(1) A finite union of σ-algebras is not necessarily a σ-algebra.

(2) If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ is an increasing sequence of sigma-algebras of a set X, then $\mathcal{F} := \bigcup_n \mathcal{F}_n$ is also a sigma-algebra.

(3) Let $\mathcal{F}$ be a sigma-algebra on X and let $T : X \mapsto Y$ be a function. Then $\mathcal{G} := \{T(A) : A \in \mathcal{F}\}$ is a sigma-algebra on Y.

(4) Let $\mathcal{F}$ be a sigma-algebra on X and let $T : Y \mapsto X$ be a function. Then $\mathcal{G} := \{T^{-1}(A) : A \in \mathcal{F}\}$ is a sigma-algebra on Y. Here $T^{-1}(A) = \{y \in Y : T(y) \in A\}$.

(5) There is no sigma-algebra with exactly 1000 elements.

Problem 5. True or false?

(1) If $T : X \mapsto Y$ and $S \subseteq 2^Y$, then $\sigma(T^{-1}(S)) = T^{-1}(\sigma(S))$.

(2) If $T : X \mapsto Y$ and $\mathcal{G} \subseteq 2^Y$ is a sigma-algebra then $T^{-1}(\mathcal{G}) := \{T^{-1}B : B \in \mathcal{G}\}$ is a sigma-algebra in X.

(3) If $T : X \mapsto Y$ and $\mathcal{F} \subseteq 2^X$ is a sigma-algebra then $T(\mathcal{F}) := \{T(A) : A \in \mathcal{F}\}$ is a sigma-algebra in Y.

(4) If X is a metric space, then the sigma algebra generated by the collection of compact subsets of X is the Borel sigma-algebra of X.

Problem 6. Let $\mathcal{F} = \sigma(S)$ where S is a collection of subsets of X. Suppose $a, b \in X$ are such that every set in S either contains both a and b or does not contain either. Then show that the same property holds for sets in $\mathcal{F}$.

Problem 7. Consider the outer Lebesgue measure λ_2^* on $\mathbb{R}^2$. Show that $[a, b] \times [c, d]$ in Lebesgue measurable by checking the Carathéodory cut condition.

Problem 8. Suppose $A \subseteq B$ are (Lebesgue) measurable subsets of R and $\lambda(A) = \lambda(B)$. Then show that any set C such that $A \subseteq C \subseteq B$ is also measurable and that $\lambda(C) = \lambda(A)$.

Problem 9. (*) Let (X, d) be a compact metric space. Fix $\alpha > 0$ and define

$$\mu_\alpha^*(A) := \liminf_{\delta \downarrow 0} \left\{ \sum_{k=1}^{\infty} \text{dia}(B_j)^\alpha : B_j \text{ are open balls in } X \text{ such that } \text{dia}(B_j) < \delta \text{ and } \bigcup_j B_j \supseteq A \right\}.$$

Show that μ_α^* is an outer measure.

When $X = \mathbb{R}$ with the usual metric, μ_α^* is the same as λ^* (the Lebesgue outer measure) when $\alpha = 1$. Further, μ_α^* is zero if $\alpha > 1$ and $\mu_\alpha^*(A) = \infty$ if $\alpha < 1$ and A is any interval.

Problem 10. Let μ^* be an outer measure on X and let $\mathcal{L}$ be the corresponding collection of measurable subsets of X. Show that $E \in \mathcal{L}$ if and only if $\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2)$ whenever $A_1 \subseteq E$ and $A_2 \subseteq E^c$.

Problem 11. Let μ^* be an outer measure on X. Suppose A_n is μ^*-measurable for all n and $E \subseteq X$. Show that (a) If $A_n \uparrow A$, show that $\mu^*(E \cap A_n) \uparrow \mu^*(A \cap E)$. (b) In general, with $A = \liminf A_n$, show that $\liminf \mu^*(E \cap A_n) \geq \mu^*(A \cap E)$.

Problem 12. For non-empty $A \subseteq \mathbb{R}_+ = [0, \infty)$, define $\mu^*(A) = \sup A$ and $\nu^*(A) = \frac{1}{\inf A}$ and $\mu^*(\emptyset) = \nu^*(\emptyset) = 0$. Show that μ^* and ν^* are outer measures. Are they metric outer measures? What are the corresponding measurable subsets?

Problem 13. Let $(X, \mathcal{F}, \mu)$ be a measure space.

(1) If $A_n, A \in \mathcal{F}$ and $A_n \uparrow A$, show that $\mu(A_n) \uparrow \mu(A)$.

(2) If $A_n, A \in \mathcal{F}$ and $A_n \downarrow A$ and $\mu(A_n) < \infty$ for some n, then show that $\mu(A_n) \downarrow \mu(A)$.

(3) Show that the second conclusion may fail if $\mu(A_n) = \infty$ for all n.

Problem 14. A measure μ on $(X, \mathcal{F})$ is said to be σ-finite if there exist $E_1, E_2, \ldots$ in $\mathcal{F}$ such that $X = \bigcup_n E_n$ and $\mu(E_n) < \infty$ for all n.

(1) Show that a σ-finite measure space has sets of arbitrarily high but finite measure.

(2) Show that a σ-finite measure has at most countably many atoms. Show that the previous assertion is false without the σ-finiteness assumption.

Problem 15. Let $\mathcal{F} = \sigma(S)$ be a sigma algebra on X. Show that for any $A \in \mathcal{F}$, there exists countably many sets $A_1, A_2, \ldots$ in S such that $A \in \sigma(\{A_1, A_2, \ldots\})$.

Problem 16. Let $\mathcal{F}$ be a sigma algebra on X and assume that $B \subseteq X$ is not in $\mathcal{F}$. Show that the smallest sigma algebra containing $\mathcal{F}$ and B is the collection of all sets of the form $(A_1 \cap B) \cup (A_2 \cap B^c)$ where $A_1, A_2 \in \mathcal{F}$.

(*) If μ is a measure on $\mathcal{F}$, can you extend it to $\mathcal{G}$ in some way? Is the extension unique?

Problem 17. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $k.\mathbb{N} = \{k, 2k, 3k, \ldots\}$. What is the sigma algebra generated by (1) the collection $k.\mathbb{N}$, $k \geq 1$? (2) the collection $p.\mathbb{N}$ as p runs over all primes?

Problem 18. Let $0 < a_n < 1$. Let $K_0 = [0,1]$ and having defined $K_0, \ldots, K_n$ where K_n is a union of 2^n pairwise disjoint closed intervals, define K_{n+1} by deleting the middle a_n *proportion* of each interval of K_n. Let $K = \bigcap_n K_n$. Then K has properties similar to the standard Cantor set (compact, has no isolated points, contains no nontrivial interval).

Show that K is measurable and find $\lambda(K)$. Show that $\lambda(K) > 0$ if and only if $\sum_n a_n < \infty$.

Problem 19. Show that there is a sequence of Riemann integrable functions $f_n : [0,1] \mapsto \mathbb{R}$ such that $\int_0^1 |f_n(x) - f_m(x)|dx \to 0$, but such that there is no Riemann integrable function f such that $\int_0^1 |f_n(x) - f(x)|dx \to 0$. [*Note:* This is the remark made in class about Riemann integrable functions not being complete in the distance $d(f,g) = \int_0^1 |f(x) - g(x)|dx$, without using the language of pseudometric etc. *Hint:* Consider the indicator of the Cantor set K in Problem 18]

Problem 20. (*) Show that any convex set in $\mathbb{R}^d$ is (Lebesgue) measurable. Is it necessarily Borel measurable?

Problem 21. If $A \subseteq \mathbb{R}$, then there exists a Lebesgue measurable set B that contains A and such that $\lambda(B) = \lambda^*(A)$. Can we choose B to be Borel measurable?

Problem 22. (*) If $A \subseteq \mathbb{R}$ is measurable and has positive Lebesgue measure, show that it contains a three-term arithmetic progression, i.e., there exist $a, b \in A$ such that $\frac{1}{2}(a+b) \in A$.

Problem 23. Let E be a measurable subset of $\mathbb{R}$. Let $L(E)$ be the supremum of all t such that there is a surjective Lip(1) function $f : E \mapsto [0,t]$. Show that $L(E) = \lambda(E)$.

Problem 24. Let $\mathcal{A}' \subseteq \mathcal{A}$ be two algebras on X that genberate the same sigma algebra (call it $\mathcal{F}$). Now suppose we have a countably additive measure μ on $\mathcal{A}$ and let μ' be the restriction of μ to $\mathcal{A}'$.

By the theorem proved in class, both μ' and mu extend to $\mathcal{F}$ as measures. Are the Carathéodory sigma algebras the same? If yes, are the extended measures equal? On the way, is the outer measure constructed from μ and μ' the same?

As a particular case, if we start with a measure μ on a sigma algebra $\mathcal{F}$, and extend it (since $\mathcal{F}$ is also an algebra), what is the extended sigma algebra? Is it $\mathcal{F}$ or is it larger?

Problem 25. Let μ be a Borel measure on $\mathbb{R}$ such that $\mu\{x\} = 0$ for each $x \in \mathbb{R}$. Show that for any $\epsilon > 0$, there exists a dense open set $U_\epsilon \subseteq \mathbb{R}$ such that $\mu(U_\epsilon) < \epsilon$.

Problem 26. Let $X = \{0,1\}^{\mathbb{N}}$ be the sequence space of zeros and ones. An element $\omega \in X$ is written as $\omega = (\omega_1, \omega_2, \ldots)$.

(1) A cylinder set is one defined by specifying the values of finitely many co-ordinates. Eg., $\{\omega : \omega_1 = 0, \omega_2 = 1, \omega_7 = 1\}$. Show that the complement of a cylinder set is a finite union of pairwise disjoint cylinder sets. Use this to describe $\mathcal{A}(S)$ as the collection of all finite unions of pairwise disjoint cylinder sets.

(2) If A is a cylinder set for which exactly n co-ordinate values are specified, define $\mu_0(A) = 2^{-n}$. Extend in the obvious way to $\mathcal{A}(S)$ and show that mu_0 is countably additive on $\mathcal{A}(S)$.

(3) Argue that there is a measure μ on $\sigma(S)$ that extends μ_0.

[*Note:* This exercise is to make precise the notion of a infinite sequence of fair coin tosses.]

Problem 27. Let A and B be measurable subsets of the line.

(1) Show that $\lambda(A+B) \geq \lambda(A) + \lambda(B)$. [*Note:* The discrete version of this is: If A and B are subsets of $\mathbb{Z}$, then $|A+B| \geq |A| + |B| - 1$, where $|A|$ denotes the cardinality of A.]

(2) Show that there can be no reverse inequality by constructing A, B having zero measure but such that $A + B = \mathbb{R}$.

Problem 28. Let A, B be measurable subsets of the real line.

(1) If A and B have positive measure, show that $A - B$ contains an interval.

(2) If $x_n + A = A$ for a sequence $x_n \to 0$, then show that $\lambda(A) = 0$ or $\lambda(A^c) = 0$.

Problem 29. Let A be a bounded subset of $\mathbb{R}$ with positive Lebesgue measure. Given are $\epsilon > 0$ and a sequence δ_n converging to zero. Show that there exists $B \subseteq A$ and a subseequnce δ_{n_k} such that (a) B is measurable and $\lambda(B) > \lambda(A) - \epsilon$, (b) if $x \in B$ then $x \pm \delta_{n_k} \in B$ for all k.

Problem 30. Let A be a measurable subset of $\mathbb{R}$. If $\lambda_1(A) > 0$, show that $(A+\mathbb{Q})^c$ has zero Lebesgue measure.

Problem 31. Show that $A \subseteq \mathbb{R}$ is measurable if and only if it is of the form $F \sqcup N$ where F is an F_σ set (a countable union of closed sets) and N is a null set (has zero outer measure).

Problem 32. If K is a compact subset of $\mathbb{R}^d$, the the set $\{x \in \mathbb{R}^d : d(x,K) = 1\}$ has zero Lebesgue measure.

Problem 33. Let A be a measurable subset of $\mathbb{R}^2$ with positive Lebesgue measure. Show that there is a measurable set B having positive measure and a number $\delta > 0$ such that $B, B + (\delta, 0), B + (0, \delta), B + (\delta, \delta)$ are all subsets of A.

Problem 34. For any $p \in (0, 1]$, construct a dense open subset of $[0, 1]$ with Lebesgue measure p. Can you do this in $\mathbb{R}$ for any $0 < p \leq \infty$?

Problem 35. Construct a Borel subset A of $\mathbb{R}$ such that $0 < \lambda(A \cap I) < \lambda(I)$ for all intervals I.

Problem 36. Let $A \subseteq \mathbb{R}$ is a bounded measurable set and let $\{x_1, x_2, \ldots\}$ be a bounded sequence. If the sets $A + x_n$, $n \geq 1$, are pairwise disjoint, then show that $\lambda(A) = 0$. What if the boundedness assumption is dropped on A or on $\{x_1, x_2, \ldots\}$?

Problem 37. Let $A \subseteq \mathbb{R}$ be a measurable set with $\lambda(A) > 0$. Show that for any $k \geq 2$, there exists a measurable $B \subseteq A$ with $\lambda(B) > 0$ and a $\delta > 0$ such that $B + i\delta$, $1 \leq i \leq k$, are pairwise disjoint.

Problem 38. If $A \subseteq \mathbb{R}^2$ is measurable, then $\lambda_2(A) = \inf \sum_n |B_n|$ where the infimum is over all countable coverings of A by open balls $B_1, B_2, \ldots$ and $|B_n|$ is the area. Deduce that Lebesgue measure on $\mathbb{R}^2$ is rotation-invariant.

[*Note:* We defined Lebesgue measure using coverings by open rectangles (with sides parallel to the axes). This exercise is to show that the shape of the basic sets does not affect the measure we get].

Problem 39. A real number is said to be *transcendental* if it is not a root of any polynomial with integer coefficients. Show that the set of transcendental numbers is a Borel set, and determine its measure.

Problem 40. A real number x is said to have exponent α for approximation by rationals if there are infinitely many rationals p/q such that $|x - \frac{p}{q}| \leq Cq^{-\alpha}$ for some $C < \infty$. Show that the set of all reals with exponent more than 2, has zero Lebesgue measure.

Problem 41. For any compact set $K \subseteq \mathbb{R}$, show that there is a Borel measure μ on $\mathbb{R}$ with $\mu(K) = 1$ and $\mu(K^c) = 0$.

Problem 42. Let μ, ν be measures on the power set of $\{0, 1\}^m$ such that $\mu\{x : x_i = 1 \ \forall i \in S\} = \nu\{x : x_i = 1 \ \forall i \in S\}$ for any $S \subseteq \{1, 2, \ldots, m\}$. Show that $\mu = \nu$.

Problem 43. True or false?

(1) If $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is continuous and $A \subseteq \mathbb{R}^2$ is measurable, then $f(A) := \{f(x) : x \in A\}$ is also measurable.

(2) Let $\mu : \mathcal{A} \mapsto [0, \infty]$ be a finitely additive set function on an algebra $\mathcal{A}$. Then countable additivity of μ is equivalent to continuity of μ under monotone limits (i.e., $A_n \uparrow A$, $A_, A \in \mathcal{A}$ implies $\mu(A_n) \uparrow \mu(A)$ and similarly for decreasing limits).

(3) $\mathcal{G} \subseteq \mathcal{F}$ are sigma algebras on X. If μ is a σ-finite measure on $\mathcal{F}$ then its restriction to $\mathcal{G}$ is also σ-finite.

Problem 44. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function. Show that any of the following conditions implies that f is Borel measurable.

(1) f is increasing.

(2) f is right-continuous.

(3) f is lower semi-continuous (means that $f(x) = \liminf\limits_{y \to x} f(y)$ for all x).

Problem 45. If $f : \mathbb{R} \mapsto \mathbb{R}$ has the property that inverse images of singleton sets are measurable, is it necessarily true that f is measurable? [Recall that on the range space $\mathbb{R}$ we always take the Borel sigma algebra. On the domain we may take either the Lebesgue or Borel sigma-algebras, and correspondingly f will be called measurable or Borel measurable]

Problem 46. Let $(X, \mathcal{F})$ be a measure space and let $f_n : X \mapsto \mathbb{R}$ be a sequence of Borel measurable functions.

(1) Show that $\sup_n f_n$, $\limsup f_n$, $\lim f_n$ (assuming limit exists), $\sum_n f_n$ (assuming that the sum converges pointwise), are measurable (where necessary, these functions may be allowed to take values $\pm\infty$ also).

(2) Show that the set $\{x \in X : \lim_n f_n(x) \text{ exists }\}$ is measurable in X. Same for $\{x : \limsup_n f_n(x) = +\infty\}$, $\{x : \lim f_n(x) = +\infty\}$.

(3) Show that the supremum of an uncountable family of measurable functions need not be measurable (even if the supremum is finite everywhere).

Problem 47. Let (X, Σ) be a measure space and suppose $f_1, f_2 : X \mapsto \mathbb{R}$. Then $F = (f_1, f_2) : X \mapsto \mathbb{R}^2$.

(1) Show that F is Borel measurable if and only if both f_1, f_2 are Borel measurable.

(2) Deduce that if f_1, f_2 are measurable, then so are $af_1 + bf_2$ (for any $a, b \in \mathbb{R}$) and $f_1 f_2$ and f_1/f_2 (for the last one, assume that $f_2 \geq 0$ so that f_1/f_2 can be defined unambiguously as an $\overline{\mathbb{R}}$-valued function).

(3) If $f : \mathbb{R} \mapsto \mathbb{R}$ is pointwise differentiable, then show that $f' : \mathbb{R} \mapsto \mathbb{R}$ is measurable.

Problem 48. Let $f : \mathbb{R} \mapsto \mathbb{R}$.

(1) If f is non-decreasing, show that it has at most countably many jumps.

(2) For any f, show that there are at most countably many $y \in \mathbb{R}$ for which $f^{-1}\{y\}$ has positive (Lebesgue) measure.

Problem 49. Find a measurable function on an appropriate interval in $\mathbb{R}$ so that the push-forward of Lebesgue measure of the interval is the measure μ satisfying

(1) $\mu(a, b] = \int_a^b e^{-|x|} dx$ for $a < b$.

(2) $\mu(a, b] = \int_a^b \frac{1}{1+x^2} dx$ for $a < b$.

(3) $\mu(a, b] =$ number of integers in $(a, b]$.

Problem 50. Let K be the $1/3$-Cantor set.

(1) Write out in detail the sketch given in class that there is a bijection $T : [0, 1] \mapsto K$ such that T and T^{-1} are Borel measurable.

(2) Use T to construct a Lebesgue measurable set in $\mathbb{R}$ that is not Borel measurable. [*Hint:* Use the existence of non-measurable sets.]

Problem 51. (*) This exercise is to show the isomorphism between $((0, 1), \mathcal{B}_{(0,1)}, \lambda_1)$ and $((0, 1)^2, \mathcal{B}_{(0,1)^2}, \lambda_2)$. As pointed out in class, the essential idea is to take a number $x = 0.x_1x_2...$ in binary expansion and map it to (y, z) where $y = 0.x_1x_3...$ and $z = 0.x_2x_4....$ But because of the ambiguities of binary expansion at dyadic rationals, this is not quite a bijection between $(0, 1)$ and $(0, 1)^2$. Here is how to fix this.

(1) Find $T_1 : (0, 1) \mapsto (0, 1)^2$ that is injective and such that $\mathrm{Im}(T_1)$ is a Borel set in $(0, 1)^2$.

(2) Find $T_2 : (0, 1)^2 \mapsto (0, 1)$ that is injective and such that $\mathrm{Im}(T_2)$ is a Borel set in $(0, 1)$.

(3) Use the idea of the proof of Schroder-Bernstein theorem to get a bijection $T : (0, 1) \mapsto (0, 1)^2$ so that T, T^{-1} are Borel measurable.

(4) If you base your T_2 on the binary expansion idea above, it turns out that Lebesgue measure on the two spaces are pushed forward to each other by T and T^{-1}.

Problem 52. (**16 marks**) State True or False and justify accordingly.

(1) If S is a collection of subsets of X, then $\mathcal{A}(S) = \bigcup_F \mathcal{A}(F)$ where the union is over all *finite* subsets $F \subseteq S$.

(2) If A is a bounded, measurable subset of $\mathbb{R}$, then $\lambda(A) = \inf\{\lambda(C) : C \supseteq A,\ C \text{ is closed}\}$.

(3) For $A \subseteq \mathbb{R}^2$, let $\Pi(A) = \{x : (x,y) \in A \text{ for some } y\}$ (a subset of $\mathbb{R}$). If A is measurable, then so is $\Pi(A)$.

(4) If A is a measurable subset of $\mathbb{R}^d$ and $0 \le x \le \lambda_d(A)$, then there is a measurable subset $B \subseteq A$ such that $\lambda_d(B) = x$.

Problem 53. (10 marks) Let $A = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, \text{ and } x + y < 1\}$. Just from the definition of the outer measure, calculate $\lambda_2(A)$.

Problem 54. (10 marks) Let S be the collection of all intervals of the form $(-a, a)$ with $a \in \mathbb{R}$. Show that if $A \in \sigma(S)$ and $x \in A$, then $-x \in A$ (in short, $A = -A$).

Problem 55. (10 marks) Let X and Y be metric spaces and let $f : X \mapsto Y$ be a continuous function. If A is a Borel set in Y, show that $f^{-1}(A)$ is a Borel set in X.

Problem 56. (10 marks) Let A and B be measurable subsets of the line with positive measure. Show that there exists some $x \in \mathbb{R}$ such that $A \cap (B + x)$ is an infinite set.

Problem 57. Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. If f is measurable, show that $f(x) = e^{ax}$ for some $a \in \mathbb{R}$. [*Remark:* Likely, you have seen this problem with the hypothesis that f is continuous. The point here is that the weaker hypothesis of measurability is enough.]

Problem 58. Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ satisfies $f(x + y) = \varphi(f(x), f(y))$ for all $x, y \in \mathbb{R}$, where $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a continuous function. If f is measurable, show that it is continuous.

Problem 59. Suppose the set of discontinuity points of $f : \mathbb{R} \mapsto \mathbb{R}$ has has zero Lebesgue measure. Show that f is a measurable function (w.r.t the Lebesgue measure).

Problem 60. Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is such that at each point, f is either right continuous or left continuous (or both). Is f necessarily Borel measurable?

Problem 61. Let f be a real valued function on $(X, \mathcal{F}, \mu)$. Show that f is Lebesgue measurable if and only if there exist Borel measurable function $g, h : X \mapsto \mathbb{R}$ such that $g \le f \le h$ and $g = h\ a.e.[\lambda]$. [*Remark:* this is the analogue of the statement that Lebesgue measurable sets are sandwiched between Borel sets of equal measure]

Problem 62. Consider an uncountable set X with the countable-co countable sigma algebra ($\mathcal{F} = \{A : A \text{ or } A^c \text{ is countable}\}$). Show that $f : X \mapsto \mathbb{R}$ is measurable if and only if there is a countable set $A \subseteq X$ such that $f\big|_{A^c}$ is constant.

Problem 63. Let $(X, \mathcal{F}, \mu)$ be a measure space. If f $f : X \mapsto \mathbb{R}$ is a bounded measurable function. Show that there exist simple functions s_n such that $s_n \to f$ uniformly on X.

Problem 64. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{S}_+$.

(1) Show that $t \mapsto \mu\{f > t\}$ is measurable from $[0, \infty]$ to $[0, \infty]$.

(2) Show that $\int_X f d\mu = \int_0^\infty \mu\{f > t\} dt$.

(3) What about $\int_0^\infty \mu\{f \geq t\} dt$?

Problem 65. Let f be a non-negative simple function on $(X, \mathcal{F}, \mu)$. Define $\nu : \mathcal{F} \mapsto [0, \infty]$ be defined by $\nu(A) = \int_X f \mathbf{1}_A d\mu$. Show that ν is a measure.

Problem 66. Let f be a non-negative simple function on $(X, \mathcal{F}, \mu)$. Show that for any $t > 0$ we have $\mu\{f \geq t\} \leq \frac{1}{t} \int_X f d\mu$ (Markov's inequality).

Problem 67. Consider $(X, \mathcal{F}, \mu)$. If f, g are two measurable functions such that $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$. Show that $f = g \ a.s.[\mu]$.

Problem 68. If f, g are two measurable functions on $([0, 1], \mathcal{L}, \lambda)$ such that $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{L}$ having $\lambda(A) = \frac{1}{3}$. Show that $f = g \ a.s.[\lambda]$.

Problem 69. Consider the space of Riemann integrable functions on $[0, 1]$ with (pseudo)-metric $d(f, g) = \int_0^1 |f - g|$. Show that this metric space is not complete.

Problem 70. Suppose f, g are integrable functions on a measure space $X, \mathcal{F}, \mu)$. Which of the following are necessarily integrable? (a) $f + g$, (b) $f - g$, (c) fg, (d) f/g, (e) $f \vee g$, (f) $f \wedge g$.

Problem 71. Given an integrable function f on $(X, \mathcal{F}, \mu)$, show that there exist simple functions s_n such that $\int_X |f - s_n| d\mu \to 0$ as $n \to \infty$.

Problem 72. (1) If f_n are non-negative measurable functions on $(X, \mathcal{F}, \mu)$ and $f = \sum_n f_n$, then show that $\int_X f d\mu = \sum_n \int_n f_n d\mu$.

(2) If g is a non-negative measurable function and $\nu(A) := \int_A g d\mu$ for $A \in \mathcal{F}$, (convention: By $\int_A f d\mu$ we just mean $\int_X f \mathbf{1}_A d\mu$), then show that ν is a measure on $\mathcal{F}$.

Problem 73. Suppose $f : (a, b) \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that (a) $\theta \mapsto f(\theta, x)$ is differentiable for each $x \in \mathbb{R}$, (b) $x \mapsto f(\theta, x)$ is integrable for each $\theta \in (a, b)$, (c) $|f(\theta_1, x) - f(\theta_2, x)| \le g(x)|\theta_1 - \theta_2|$ for all $\theta_1, \theta_2 \in (a, b)$ and $x \in \mathbb{R}$ and that g is integrable over $\mathbb{R}$ (w.r.t. Lebesgue measure). Define $H(\theta) := \int_{\mathbb{R}} f(\theta, x) d\lambda(x)$.

Show that H is differentiable, that $\frac{d}{d\theta} f(\theta, x)$ is integrable over $\mathbb{R}$ for each θ, and that

$$\frac{d}{d\theta} H(\theta) = \int_{\mathbb{R}} \frac{d}{d\theta} f(\theta, x) d\lambda(x).$$

Problem 74. Assume that f_n are non-negative measurable functions that are bounded above by an integrable function g a.e.[μ] (i.e. $0 \le f_n \le g$ a.e.[μ]).

(1) Show that $\limsup\limits_{n\to\infty} \int_X f_n d\mu \le \int_X (\limsup\limits_{n\to\infty} f_n) d\mu$.

(2) If $f_n \downarrow f$ a.e.[μ], and $\int_X f_n d\mu$ is finite for some n, then show that $\int_X f_n d\mu \downarrow \int_X f d\mu$.

Problem 75. If f is an integrable function, show that $\int |f| \mathbf{1}_{|f|>n} d\mu \to 0$. More generally, if A_n is any sequence of events such that $\mu(A_n) \to 0$, then show that $\int_{A_n} |f| d\mu \to 0$ as $n \to \infty$.

Problem 76. If $f : [a, b] \mapsto \mathbb{R}$ is a continuous function, show that its Riemann integral is equal to its Lebesgue integral (w.r.t. Lebesgue measure on $[a, b]$). [*Note:* This is true more generally.]

Problem 77. On $[0, 1]$ and $[0, \infty)$ (both endowed with Lebesgue measure), for which α, β are x^α and $x^\alpha (\log x)^\beta$ are integrable.

Problem 78. Let A_n be measurable sets in $(X, \mathcal{F}, \mu)$. Let A be the set of all x that belong to A_n for infinitely many n (is A measurable?).

(1) If $\sum_n \mu(A_n) < \infty$, then show that $\mu(A) = 0$.

(2) Show that $\mu(A_n) \to 0$ does not necessarily imply that $\mu(A) = 0$.

Problem 79. Let f be a non-negative integrable function on $(X, \mathcal{F}, \mu)$. Let $I_n(p) = \int_X n \log(1 + \frac{f^p}{n^p})$ for $p > 0$. Show that $I_n(p) \to \int_X f d\mu$ as $n \to \infty$ for $p = 1$. What happens for $p < 1$ and for $p > 1$?

Problem 80. If f_n are non-negative measurable functions with $\int_X f_n d\mu = 1$ for all n and $f_n \to f$ a.e.[μ], then show that the set of all possible values of $\int_X f d\mu$ is $[0, 1]$.

Problem 81. Suppose f_n, f are non-negative integrable functions on $(X, \mathcal{F}, \mu)$ and that $f_n \to f$ a.s.$[\mu]$. Show that $\int_X f_n d\mu - \|f_n - f\|_{L^1(\mu)} \to \int_X f d\mu$.

Problem 82. State whether true or false and justify. Whenever we write L^p, it is assumed that $1 \le p \le \infty$.

(1) $L^\infty(\mathbb{R}, \mathcal{B}, \lambda_1)$ is not separable (has no countable dense subset).

(2) If $p_1 < p_2$, then $L^{p_1}(X, \mathcal{F}, \mu) \supseteq L^{p_2}(X, \mathcal{F}, \mu)$.

(3) If $p_1 < p_2 < p_3$ then $L^{p_1}(\mu) \cap L^{p_3}(\mu) \subseteq L^{p_2}(\mu)$.

(4) If μ is a finite measure, then for any measurable f, the set $\{p \ge 1 : f \in L^p(\mu)\}$ is an interval of the form $[1, r]$ or $[1, r)$ for some $r \le \infty$.

(5) If f is a measurable function on $(X, \mathcal{F}, \mu)$, then the set $\{p \ge 1 : f \in L^p(\mu)\}$ is an interval (Extra: Give exmples to show that this interval can be open or closed on the left or right).

(6) $L^1(\mu) \cap L^2(\mu)$ is complete in the norm $\|\cdot\|_1 + \|\cdot\|_2$.

Problem 83. Show that $L^\infty(X, \mathcal{F}, \mu)$ is complete (This was skipped in class).

Problem 84. Let $f \in L^p(X, \mathcal{F}, \mu)$ with $p < \infty$.

(1) Show that $\|f\|_p^p = \int_0^\infty p t^{p-1} \mu\{|f| > t\} dt$. [*Note:* This was shown for $p = 1$ and simple non-negative f in an earlier exercise.]

(2) Show that that if $f \in L^p(\mu)$, then $\mu\{|f| > t\} \le \|f\|_p^p t^{-p}$.

Problem 85. Let f be a non-negative measurable function on $(X, \mathcal{F}, \mu)$.

(1) Show that $\mu\{f > 0\} \ge \frac{(\int_X f d\mu)^2}{\int_X f^2 d\mu}$.

(2) Show that no non-trivial lower bound for $\mu\{f > 0\}$ can be obtained in terms of $\int_X f d\mu$ alone.

Problem 86. Let μ be a finite measure. Then show that the following are equivalent.

(1) $f \in L^\infty(\mu)$.

(2) $f \in L^p(\mu)$ for all $p < \infty$ and $\sup_p \|f\|_p < \infty$.

If these equivalent conditions hold, show that $\|f\|_p \to \|f\|_\infty$.

Problem 87. Let f be a non-negative measurable function on $(X, \mathcal{F}, \mu)$ with $\int_X f d\mu = 1$. Show that if $A \in \mathcal{F}$ with finite positive measure, then

(1) $\int_X f^p d\mu \le \mu(A)^{1-p}$ if $0 < p < 1$.

(2) $\int_X \log f \, d\mu \leq \mu(A) \log \frac{1}{\mu(A)}$.

Problem 88. (1) If $f, g \in L^2$, show that $\|f-g\|^2 + \|f+g\|^2 = 2\|f\|^2 + 2\|g\|^2$.

(2) (*) If $f_1, \ldots, f_n \in L^2(\mu)$, show that $(\langle f_i, f_j \rangle_{L^2(\mu)})_{1 \leq i,j \leq n}$ is a positive semi-definite matrix.

(3) Give examples to show that $L^p(X, \mathcal{F}, \mu)$ is not a Hilbert space if $p \neq 2$.

Problem 89. Let μ be a probability measure on X. If f, g are non-negative measurable and $fg \geq 1$ a.e.$[\mu]$, then $\left(\int_X f d\mu\right)\left(\int_X g d\mu\right) \geq 1$.

Problem 90. Suppose μ_n is a sequence of Radon measures on $\mathbb{R}^d$ such that $\lim_{n\to\infty} \int f d\mu_n$ exists and is finite for every $f \in C_c(\mathbb{R}^d)$. Then show that there is a Radon measure μ such that $\int f d\mu_n \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^d)$.

Problem 91. Let $1 \leq p \leq \infty$. Is $L^p(\mathbb{R}, \mathcal{B}, \lambda_1)$ separable in L^p metric?

Problem 92. Suppose $f_n \in L^1(\mu) \cap L^2(\mu)$. If $f_n \to g$ in $L^1(\mu)$ and $f_n \to h$ in $L^2(\mu)$, then $g = h$ a.e.$[\mu]$.

Problem 93. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measure spaces such that $\nu = \mu \circ T^{-1}$ for some $T : X \mapsto Y$. Then show that for any $f \in L^1(\nu)$, the function $f \circ T \in L^1(\mu)$ and that $\int_X (f \circ T) d\mu = \int_Y f d\nu$.

Problem 94. If $f \in L^p[0,1]$ and $\int fg \, d\lambda = 0$ for any $g \in L^q[0,1]$ where $\frac{1}{p} + \frac{1}{q} = 1$, then show that $f = 0 \; a.e.$

Problem 95. If $f \in L^p[0,1]$, for which values of α is it necessarily true that $\frac{1}{\epsilon^\alpha} \int_{[0,\epsilon]} f \, d\lambda \to 0$ as $\epsilon \to 0$? Doing it for $p = 2$ first may suggest the answer.

Problem 96. Let L be a positive linear functional on $C_c^\infty(\mathbb{R}^d)$ (endowed with sup-norm). Show that $L(f) = \int_X f d\mu$ for a unique Radon measure μ on $\mathbb{R}^d$.

Problem 97. Which of the following sets is dense in $L^p([0,1], \mathcal{B}, \lambda_1)$? Consider the case $p = \infty$ carefully.

(1) $C[0,1]$.

(2) $C^\infty[0,1]$.

(3) The set of all polynomials.

(4) The collection of all step functions.

Problem 98. If μ is a finite measure, show that $\frac{\|f\|_{p+1}^{p+1}}{\|f\|_p^p} \to \|f\|_\infty$ as $p \to \infty$.

Problem 99. Suppose $0 < p < 1$. If f, g are non-negative measurable function on $(X, \mathcal{F}, \mu)$, then $\left(\int_X (f+g)^p\right)^{1/p} \geq \left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}$.

Problem 100. Let D be a bounded open set in the complex plane and let μ be the Lebesgue measure on D. Let H be the collection of all holomorphic functions on $\mathbb{D}$ that are in $L^2(D)$. Show that H is a closed subspace in L^2 norm and hence a Hilbert space itself.

Problem 101. Suppose $1 \leq p < q < r < \infty$. If $f_n \in L^p(\mu) \cap L^r(\mu)$ and this sequence is Cauchy in $L^p(\mu)$ and Cauchy in $L^r(\mu)$, then f_n converges in $L^q(\mu)$.

Problem 102. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $A_n \in \mathcal{F}$. Write out explicitly the meaning of $\mathbf{1}_{A_n} \to 0$ in $a.e.[\mu]$ sense, in measure and in L^1. Which of these imply the others? (Do this directly, without invoking the general theorems proved in class). What if μ is finite?

Problem 103. Let μ be a measure that is not supported at a single point. Show that $L^p(\mu)$ norm does not come from an inner product if $p \neq 2$ and does come from an inner product if $p = 2$.

Problem 104. Suppose μ is a Borel measure on $[0, 1]$ such that $\int fg d\mu = \left(\int f d\mu\right)\left(\int g d\mu\right)$ for all$f, g \in C[0, 1]$. Show that $\mu = \delta_a$ for some $a \in [0, 1]$.

Problem 105. Suppose f_n, f are non-negative measurable functions such that $f_n \to f$ $a.e.[\mu]$. Show that $\int_X f_n d\mu \to \int_X f d\mu$ if and only if $\int_X |f_n - f| d\mu \to 0$.

Problem 106. If $f_n \to f$ in measure μ, and $|f_n| \leq g$ for some integrable function g, then show that $\int_X |f_n - f| d\mu \to 0$ (DCT under convergence in measure only).

Problem 107. Let f_n, f be measurable functions on $(X, \mathcal{F}, \mu)$. If $\mathbf{1}_{|f_n - f| \geq \delta} \to 0$ a.e.$[\mu]$ for every $\delta > 0$, then is it true that $f_n \to f$ a.e.$[\mu]$?

Problem 108. (**16 marks**) State True or False and justify accordingly.

(1) If $f \in L^1(\mu) \cap L^3(\mu)$, then $f \in L^2(\mu)$.

(2) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \mapsto \mathbb{R}$ be an integrable function. Let $A_n \in \mathcal{F}$ be a sequence of measurable sets. If $\mu(A_n) \to 0$ as $n \to \infty$, then $\int_{A_n} f d\mu \to 0$.

(3) On a measure space $(X, \mathcal{F}, \mu)$ suppose $f_n \to f$ $a.e.[\mu]$ and $\int f_n d\mu \to \int f d\mu$. Then there is an integrable function g such that $|f_n| \leq g$ $a.e.[\mu]$, for all n.

(4) Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined as $f(x, y) = x - y$ and let $\mathcal{F}$ be the smallest σ-algebra on $\mathbb{R}^2$ such that f is Borel measurable. Then $\mathcal{F}$ does not contain any non-empty bounded set in the plane.

Problem 109. (10 marks) If $f : \mathbb{R} \mapsto \mathbb{R}$ is right-continuous, then f is Borel measurable.

Problem 110. (10 marks) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \mapsto \mathbb{R}_+$ be a non-negative measurable function. Define $\theta(A) = \int_A f d\mu$ for $A \in \mathcal{F}$.

(1) Show that θ is a measure on $\mathcal{F}$.

(2) Show that $g : X \mapsto \mathbb{R}$ is integrable w.r.t. θ if and only if gf is integrable w.r.t. μ and $\int_X g d\theta = \int_X gf d\mu$.

Problem 111. (10 marks) Suppose $1 < p_1 < p_2 < p_3 < \infty$ are such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. If f_1, f_2, f_3 are measurable functions on $(X, \mathcal{F}, \mu)$ such that $f_i \in L^{p_i}(\mu)$ for $i = 1, 2, 3$, then show that $f_1 f_2 f_3 \in L^1$ and $\|f_1 f_2 f_3\|_1 \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3}$.

Problem 112. (10 marks) Let $(X, \mathcal{F}, \mu)$ be a measure space and let f be a non-negative measurable function.

(1) If f takes only integer values, then show that $\int f d\mu = \sum_{n=1}^{\infty} \mu\{f \geq n\}$.

(2) In general (i.e., for non-negative measurable f), show that f is integrable if and only if $\sum_{n=1}^{\infty} \mu\{f \geq n\}$ converges.

Problem 113. If f is a measurable function on $(X, \mathcal{F}, \mu)$ such that $\int f d\mu = \int f^2 d\mu = \int f^4 d\mu$ (all integrals assumed to exist and finite). Then $f = \mathbf{1}_A$ for some $A \in \mathcal{F}$.

Problem 114. If f is a non-negative measurable function such that $\log f$ is integrable, then

(1) $\int \frac{f^p - 1}{p} d\mu \to \int \log f d\mu$ as $p \downarrow 0$.

(2) $\frac{1}{p} \log \left(\int f^p d\mu \right) \to \int \log f d\mu$ as $p \downarrow 0$.

Problem 115. Suppose $f_n \to f$ $a.e.[\mu]$ and that $\sup_n \int f_n^2 d\mu < \infty$. Then $\int |f_n - f| d\mu \to 0$.

Problem 116. If $f_n \to f$ $a.e.[\mu]$ in $(X, \mathcal{F}, \mu)$, then show that there exist $a_n \uparrow \infty$ such that $a_n X_n \to 0$ $a.s.[\mu]$.

Problem 117. The space is $(X, \mathcal{F}, \mu)$.

(1) $L^2(\mu)$ is finite dimensional if and only if there is a finite set $F \subseteq X$ such that $\mu(F^c) = 0$.

(2) Let $1 \le p_1, \ldots, p_k \le \infty$ be such that $\frac{1}{p_1} + \ldots + \frac{1}{p_k} = \frac{1}{r}$. Show that $\|f_1 \ldots f_k\|_r \le \prod_{j=1}^k \|f_j\|_{p_j}$ for any $f_i \in L^{p_i}$.

(3) If $1 \le p < r < q \le \infty$, we can write $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ for a suitable $\alpha \in (0,1)$. Show that for any $f \in L^p \cap L^q$ we have $\|f\|_r \le \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}$. In particular, $L^p \cap L^q \subseteq L^r$.

(4) If $1 \le p < q \le \infty$, give an example to show that in general neither of L^p and L^q is contained in the other. Show that $L^p(\mu) \subseteq L^q(\mu)$ if and only if μ is finite.

(5) If $1 \le p \le 2$, show that any $f \in L^p$ can be written as $f = g + h$ with $g \in L^1$ and $h \in L^2$.

(6) Show that if $0 < p < 1$, then $L^p = \{[f] : \int_X |f|^p d\mu < \infty\}$ is a vector space but that $\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}$ is not a norm. [*Hint:* Consider the counting measure on two points]

(7) If $f_n \to f$ in L^p (where $1 \le p \le \infty$), show that there is some subsequence $\{n_k\}$ such that $f_{n_k} \to f$ $a.s.[\mu]$. [*Hint:* Recall the proof of completeness of L^p]

(8) True or False? If $f_n \to g$ in L^p and $f_n \to h$ in L^q, where $1 \le p \le q \le \infty$, then $g = h$ $a.s.[\mu]$.

Problem 118. Consider $([0,1], \mathcal{B}, \lambda)$.

(1) Find a subspace of L^2 that is dense in L^p for $p < 2$ but not dense in L^2.

(2) Find a subspace of L^∞ that is dense in L^2 but not in L^p for any $p > 2$.

Problem 119. Suppose $f_n \to f$ $a.s.[\mu]$. In this problem and the next, we investigate conditions that ensure that $f_n \to f$ in L^1.

(1) If $\sup_n \int |f_n|^p d\mu < \infty$ for some $p > 1$, show that $f_n \to f$ in L^1. Show that this fails if $p = 1$.

(2) If $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an increasing function and we replace the condition $\sup_n \int |f_n|^p d\mu < \infty$ by $\sup_n \int \varphi(|f_n|) d\mu < \infty$, what condition on φ will ensure that the conclusion remains valid?

Problem 120. Suppose $f_n \to f$ $a.s.[\mu]$. We say that the collection $\{f_n\}$ is *uniformly integrable* if given $\epsilon > 0$, there exists $M < \infty$ such that $\sup_n \int_{|f_n| > M} |f_n| d\mu < \epsilon$. Show that $f_n \to f$ in L^1 if and only if $\{f_n\}$ is uniformly integrable.

This exercise is to show that the other conditions under which almost sure convergence can be converted to convergence in L^1 are stronger than uniform integrability.

(1) If $\sup_n |f_n| \leq g \; a.s.[\mu]$ and $\int |g| d\mu < \infty$ (the conditionof DCT), then show that $\{f_n\}$ is uniformly integrable.

(2) If $\sup_n \|f_n\|_p < \infty$ for some $p > 1$, show that $\{f_n\}$ is uniformly integrable.

(3) Show that the converse is false in both statements.

Problem 121. Let X and Y be separable metric spaces and let $Z = X \times Y$ endowed with the product topology.

(1) If X and Y are separable, show that $\mathcal{B}_X \times \mathcal{B}_Y = \mathcal{B}_Z$.

(2) Suppose $X = Y$ is not separable. Then show that $D = \{(x, x) : x \in X\}$ is in $\mathcal{B}_Z$ but not in $\mathcal{B}_X \times \mathcal{B}_Y$.

Problem 122. Let μ be a Radon measure on $\mathbb{R}$. Suppose convergence in measure w.r.t. μ implies convergence almost everywhere w.r.t. μ. What can you say about μ?

Problem 123. Let $\mathcal{L}_d$ denote the Lebesgue sigma algebra on $\mathbb{R}^d$. Show that $\mathcal{L}_2 \neq \mathcal{L}_1 \times \mathcal{L}_1$.

Problem 124. let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $f(x, y) = \sin(x)\mathbf{1}_{y<x<y+2\pi}$. Show that $\iint f(x, y) dy dx \neq \iint f(x, y) dx dy$. Does this indicate a fatal flaw in Fubini's theorem as presented in thousands of books?

Problem 125. Let A be a Borel set in $\mathbb{R}^2$ such that its intersection with each vertical line is a finite set. Show that for $a.e.\; y[\lambda_1]$, the intersection of A with the horizontal line through $(0, y)$ has zero Lebesgue measure (in one dimension).

Problem 126. Suppose A, B are measurable subsets of $[0, 1)$ (treated as a group with addition modulo 1). Show that there exists some $x \in [0, 1)$ such that $\lambda(A \cap (B + x)) \geq \lambda(A)\lambda(B)$.

Problem 127. Exhibit a product measure space where the iterated integral $\int_X \int_Y f(x, y) d\nu(y) d\mu(x)$ and $\int_Y \int_X f(x, y) d\nu(y) d\mu(x)$ exist and are equal, but f is not integrable w.r.t. $\mu \otimes \nu$.

Problem 128. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a non-negative measurable function. Show that $\int_{\mathbb{R}} f(x) d\lambda(x)$ is equal to the area (two-dimensional Lebesgue measure) of $\{(x, y) : 0 \leq y \leq f(x)\}$ (the region between the graph of f and the x-axis.).

Problem 129. If f be a non-negative measurable function on $(X, \mathcal{F}, \mu)$. Show that $\int_X f d\mu = \int_0^\infty \mu\{f > t\} dt$. [*Hint:* Use Fubini's theorem on $X \times \mathbb{R}_+$ with ...]

Problem 130. Let $A \subseteq \mathbb{R}$ be a measurable set with $\lambda_1(A) = 1$. Find $\lambda_2\{(x, y) \in \mathbb{R}^2 : (1 + x^2)(y - e^x) \in A\}$.

Problem 131. Let f_n be measurable functions on $(X, \mathcal{F}, \mu)$ such that $f = \sum_n f_n$ converges $a.e.[\mu]$. From Fubini's and Tonelli's theorem discuss when the identity $\int f d\mu = \sum_n \int f_n d\mu$ holds. Consider the case when f_n are non-negative separately. How does this compare to the conditions required by MCT and DCT?

Problem 132. If $(X_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2, 3$, are σ-finite measure spaces. Show that

$$(\mathcal{F}_1 \times \mathcal{F}_2) \times \mathcal{F}_3 = (\mathcal{F}_1 \times \mathcal{F}_2) \times \mathcal{F}_3 \quad \text{and} \quad (\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3).$$

This justifies writing $\mu_1 \times \mu_2 \times \mu_3$ etc.

Problem 133. $A \subseteq \mathbb{R}^2$ is such that for each line L, the set $A \cap L$ is countable. Then $\lambda_2(A) = 0$. Can you do the same if the hypothesis is given only for lines that pass through the origin?

Problem 134. Write $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^2 \setminus \{0\} = (0, \infty) \times [0, 2\pi)$ with the identification $(x, y) \leftrightarrow (r, \theta)$ if $x = r \cos\theta$ and $y = r \sin\theta$ (polar co-ordinates). Let λ denote the one-dimensional Lebesgue measure on $\mathbb{R}$. Let λ_+ and λ_0 denote its restriction to $(0, \infty)$ and to $[0, 2\pi)$.

(1) Show that on $\mathbb{R}^2$ (or $\mathbb{R}^2 \setminus \{0\}$), the two measures $\lambda \times \lambda$ and $\lambda_+ \times \lambda_0$ are equal.
(2) Deduce the integration formula $\int_\mathbb{R}\int_\mathbb{R} f(x, y)\, dx\, dy = \int_{(0,\infty)}\int_{[0,2\pi)} f(r\cos\theta, r\sin\theta) r d\theta dr$ for $f \in L^1(\lambda_2)$.

Problem 135. Use Fubini's theorem (state the setting and check the conditions each time!) to deduce the following statements that we have seen before.

(1) If $f_n \geq 0$ are measurable, then $\int_X \left(\sum_n f_n\right) d\mu = \sum_\int f_n d\mu$.
(2) If $f \geq 0$ is measurable, then $\int_X f d\mu = \int_0^\infty \mu\{f > t\} dt$.
(3) If $f \geq 0$ is measurable, then $\int_X f^p d\mu = p \int \mu\{f > t\} t^{p-1} dt$.

Problem 136. Let $f : [0, 1] \mapsto \mathbb{R}$ be measurable. Show that f is integrable w.r.t. Lebesgue measure on $[0, 1]$ if and only if $g(x, y) := f(x) - f(y)$ is integrable w.r.t. Lebesgue measure on $[0, 1]^2$.

Problem 137. Let μ, ν be Borel probability measures on $\mathbb{R}$. Show that $(x+y)^2$ is integrable w.r.t. $\mu \times \nu$ if and only if x^2 is integrable w.r.t. μ and w.r.t. ν.

Problem 138. Let $f : \mathbb{R} \mapsto \mathbb{R}_+$ be a non-negative function and let $A_f = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le f(x)\}$.

(1) Show that f is measurable if and only if A_f is a measurable set in $\mathbb{R}^2$.

(2) Assume that f is measurable. Then $\int f d\lambda = \lambda_2(A_f)$.

Problem 139. Let μ_n be a sequence of regular Borel measures on $\mathbb{R}$ such that $\lim_{n\to\infty} f d\mu_n$ exists and is finite for every $f \in C_c(\mathbb{R})$. Show that there is a regular Borel measure μ on $\mathbb{R}^d$ such that $\int f d\mu_n \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^d)$.

Problem 140. Let μ_n, μ be regular Borel measures on $\mathbb{R}$ such that $\lim_{n\to\infty} f d\mu_n = \int f d\mu$ for every $f \in C_c(\mathbb{R})$. Then show that $\mu_n(a,b) \to \mu(a,b)$ for all except atmost countably many a, b.

Problem 141. Show that there exist probability measures μ_n, ν_n on $[0,1]$ such that $\int f d\mu_n - \int f d\nu_n \to 0$ as $n \to \infty$, for any $f \in C[0,1]$ but such that $\mu_n[0,t] - \nu_n[0,t] \not\to 0$ for any $t \in (0,1)$.

Problem 142. Let μ, ν be finite measures on $(X, \mathcal{F})$. Show that $\nu \ll \mu$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that $\nu(A) < \epsilon$ for any $A \in \mathcal{F}$ with $\mu(A) < \delta$.

Problem 143. Let f be a measurable function such that $f > 0\ a.s.[\mu]$. Show that $\inf\{\int_A f d\mu : \mu(A) \ge \epsilon\} > 0$ for any $\epsilon > 0$.

Problem 144. Construct a sigma-finite measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\mu \ll \lambda_1$ and $\mu(a,b) = \infty$ for any $a < b$.

Problem 145. Let $(X, \mathcal{F}, \mu)$ be a probability space. Let $A_n \in \mathcal{F}$ and let $A = \limsup A_n$.

(1) If $\sum_n \mu(A_n) < \infty$, then show that $\mu(A) = 0$.

(2) If $\mu(A_n) \ge \epsilon > 0$ for all n, then show that $\mu(A) \ge \epsilon$.

Problem 146. Consider $([0,1], \mathcal{B}, \lambda)$. For $\alpha > 1$, define $T_\alpha : [0,1] \mapsto \mathbb{R}$ by $T_\alpha(x) = \sum_{n\ge1} x_n alp^{-n}$ where $x = \sum_{n\ge1} x_n 2^{-n}$ is the binary expansion of x (convention: if there are two binary expansions, choose the one with infinitely many ones). Let $\mu_\alpha = \lambda \circ T_\alpha^{-1}$ be the pushforward measure. For $\alpha > 2$, show that $\mu_\alpha \perp \lambda$.

Problem 147. Let $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ be a non-negative integrable function. Show that there exists a radial function $g : \mathbb{R}^d \mapsto \mathbb{R}$ (this means that $g(x) = g(y)$ if $|x| = |y|$) such that $\lambda_d\{f > t\} = \lambda_d\{g > t\}$ for all $t > 0$. Deduce that $\int f d\lambda_d = \int g d\lambda_d$. [*Note:* g is called the spherical rearrangement of f.]

Problem 148. On $(X, \mathcal{F}, \mu)$, suppose f is a non-negative integrable function. Let

$$I(\lambda) = \sum_{n \in \mathbb{Z}} \lambda^n \mu\{\lambda^n \leq f < \lambda^{n+1}\}.$$

Show that $I(\lambda)$ is well-defined (i.e., the series converges absolutely) for every $\lambda > 1$ and that $I(\lambda) \downarrow \int f d\mu$ as $\lambda \downarrow 1$.

Problem 149. Let f be a non-negative measurable function on the sigma-finite measure space $(X, \mathcal{F})$ and let $\nu(A) = \int_A f d\mu$. Show that ν is sigma-finite.

Problem 150. Let f be a non-negative measurable function on the sigma-finite measure space $(X, \mathcal{F})$. If $\mathcal{G}$ is a sub-sigma algebra of $\mathcal{F}$, then show that there exists a $g : X \mapsto \mathbb{R}_+$ that is Borel measurable w.r.t. $\mathcal{G}$ and such that $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{G}$.

Problem 151. Suppose $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ (all sigma-finite measure on some measure spaces). Then show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

Problem 152. (**16 marks**) State "True" or "False" and justify accordingly.

(1) Let $\mathcal{F}$ be the collection of all those Borel subsets of $\mathbb{R}$ that have zero Lebesgue measure or whose complement has zero Lebesgue measure. Then $\mathcal{F}$ is a σ-algebra.

(2) If f is an integrable function on $(X, \mathcal{F}, \mu)$, then $t\mu\{f > t\} \to 0$ as $t \to \infty$.

(3) Suppose A_n are subsets of $[0, 1]$ such that $\lambda(A_n) \leq \frac{1}{2}$ for all n. Then $\lambda(\limsup A_n) \leq \frac{1}{2}$.

(4) Suppose μ and ν are measures on $(X, \mathcal{F})$ such that ν is absolutely continuous to μ. If μ is a finite measure, then so is ν.

(5) For $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, let $A_k = \{k, 2k, 3k, \ldots\}$. Then the sigma algebra generated by the sets A_p as p varies over all primes, is the power set of $\mathbb{N}$.

Problem 153. (**16 marks**)

(1) Let $\mathcal{F}$ be the sigma-algebra on $\mathbb{R}$ generated by the intervals $(0, 2)$ and $(1, 3)$. What is the cardinality of $\mathcal{F}$?

(2) Let μ^* be an outer measure on$X$. Suppose $A_1, B_1, A_2, B_2, \ldots$ are subsets of X such that $\mu^*(A_n \Delta B_n) = 0$ for each $n \geq 1$. Show that $\mu^*(\cup_n A_n) = \mu^*(\cup_n B_n)$.

(3) Suppose $A_1, \ldots, A_9$ are Borel subsets of $[0,1]$ such that each $x \in [0,1]$ belongs to at least three distinct sets among $A_1, \ldots, A_9$. Show that $\lambda(A_k) \geq \frac{1}{3}$ for some k.

(4) Let $A \subseteq \mathbb{R}$ be a Borel set with $\lambda_1(A) = 1$. Find $\lambda_2\{(x,y) \in \mathbb{R}^2 : (1+x^2)(y - e^x) \in A\}$.

Problem 154. (**6 marks**) Suppose f_n, f, g are measurable functions on $(X, \mathcal{F}, \mu)$.

(1) If $\mathbf{1}_{|f_n - f| \geq \delta} \to 0$ a.e.$[\mu]$ for every $\delta > 0$, then show that $f_n \to f$ a.e.$[\mu]$.

(2) Assume that f and g are integrable. If $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e.$[\mu]$.

Problem 155. (**6 marks**) Given $f : \mathbb{R}^2 \mapsto \mathbb{R}$, define $g : \mathbb{R} \mapsto \mathbb{R}$ by $g(x) = f(x, 1-x)$.

(1) Take the Borel sigma algebras on $\mathbb{R}$ and $\mathbb{R}^2$. If f is measurable, then so is g.

(2) Take the Lebesgue sigma algebras on $\mathbb{R}$ and $\mathbb{R}^2$. Show that g need not be measurable even if f is measurable.

Problem 156. (**6 marks**) Consider $\mathbb{R}$ with the Borel sigma algebra.

(1) Let $f, g : [0,1] \mapsto \mathbb{R}$ be Borel measurable functions. If $h(x,y) = f(x) + g(y)$ is integrable over $[0,1]^2$, then f and g are integrable over $[0,1]$.

(2) If $f : \mathbb{R} \mapsto \mathbb{R}$ is integrable, then $\sum\limits_{n \in \mathbb{Z}} f(x+n)$ converges for a.e. x (w.r.t. Lebesgue measure).

Problem 157. (**5 marks**) If $f : \mathbb{R} \mapsto \mathbb{R}$ is integrable and $g(x) = \int\limits_{[x,x+1]} f d\lambda$, then show that g is a continuous function.

Problem 158. (**5 marks**) Suppose μ is a Borel measure on $\mathbb{R}$ such that $\mu(A) = 1$ whenever A is a Borel set with $\lambda(A) = 1$. Show that $\mu = \lambda$.